

CENTRE DE PHYSIQUE THEORIQUE**CNRS - Luminy, Case 907****13288 Marseille Cedex 9****FORCES FROM NONCOMMUTATIVE GEOMETRY****Thomas SCHÜCKER ¹****Abstract**

Einstein derived general relativity from Riemannian geometry. Connes extends this derivation to noncommutative geometry and obtains electro-magnetic, weak and strong forces. These are pseudo forces, that accompany the gravitational force just as in Minkowskian geometry the magnetic force accompanies the electric force. The main physical input of Connes' derivation is parity violation. His main output is the Higgs boson which breaks the gauge symmetry spontaneously and gives masses to gauge and Higgs bosons.

Einstein déduit la gravitation à partir de la géométrie Riemannienne. Connes étend cette dérivation à la géométrie non commutative et obtient les forces électro-magnétique, faible et forte. Ce sont des pseudo forces qui accompagnent la force gravitationnelle, au même titre qu'en géométrie Minkowskienne la force magnétique est une pseudo force qui accompagne la force électrique. L'input physique de la dérivation de Connes est la violation de la parité. Son résultat principal est le scalaire de Higgs qui brise la symétrie de jauge spontanément et rend massifs les bosons de jauge et de Higgs.

PACS-92: 11.15 Gauge field theories

MSC-91: 81T13 Yang-Mills and other gauge theories

CPT-01/P.4245

hep-th/yymmxxx

¹ and Université de Provence, schucker@cpt.univ-mrs.fr

Still today one of the major summits in physics is the understanding of the spectrum of the hydrogen atom. The phenomenological formula by Balmer and Rydberg was a remarkable pre-summit on the way up. The true summit was reached by deriving this formula from quantum mechanics. We would like to compare the standard model of electro-magnetic, weak and strong forces with the Balmer-Rydberg formula and review the present status of Connes' derivation of this model from noncommutative geometry. This geometry extends Riemannian geometry and Connes' derivation is a natural extension of another major summit in physics: Einstein's derivation of general relativity from Riemannian geometry. Indeed, Connes' derivation unifies gravity with the other three forces.

atoms	particles and forces
Balmer-Rydberg formula	standard model
quantum mechanics	noncommutative geometry

Table 1: An analogy

Let us briefly recall four nested, analytic geometries and their impact on our understanding of forces and time, see table 2. *Euclidean geometry* is underlying Newton's mechanics as space of positions. Forces are described by vectors living in the same space and the Euclidean scalar product is needed to define work and potential energy. Time is not part of geometry, it is absolute. This point of view is abandoned in special relativity unifying space and time into *Minkowskian geometry*. This new point of view allows to derive the magnetic field from the electric field as a pseudo force associated to a Lorentz boost. Although time has become relative, one can still imagine a grid of synchronized clocks, i.e. a universal time. The next generalization is Riemannian geometry = curved spacetime. Here gravity can be viewed as the pseudo force associated to a uniformly accelerated coordinate transformation. At the same time universal time loses all meaning and we must content ourselves with proper time. With today's precision in time measurement, this complication of life becomes a bare necessity, e.g. the global positioning system (GPS).

Our last generalization is to Connes' noncommutative geometry = curved space(time) with uncertainty. It allows to understand some Yang-Mills and some Higgs forces as pseudo-forces associated to transformations, that extend the two coordinate transformations above to the new geometry without points. Also, proper time comes with an uncertainty. This uncertainty of some hundred Planck times might be accessible to experiments through gravitational wave detectors within the next ten years [1].

geometry	force	time
Euclidean	$E = \int \vec{F} \cdot d\vec{x}$	absolute
Minkowskian	$\vec{E}, \epsilon_0 \Rightarrow \vec{B}, \mu_0 = \frac{1}{\epsilon_0 c^2}$	universal
Riemannian	Coriolis \leftrightarrow gravity	proper, τ
noncommutative	gravity \Rightarrow YMH, $\lambda = \frac{1}{3}g_2^2$	$\Delta\tau \sim 10^{-40}$ s

Table 2: Four nested analytic geometries

1 Slot machines and the standard model

Today we have a very precise phenomenological description of electro-magnetic, weak and strong forces. This description, the standard model, works on a perturbative quantum level and, as classical gravity, it derives from an action principle. Let us introduce this action by analogy with the Balmer-Rydberg formula.

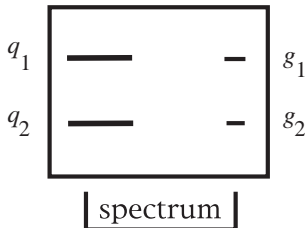


Figure 1: A slot machine for atomic spectra

One of the new features of atomic physics was the appearance of discrete frequencies and the measurement of atomic spectra became a highly developed art. It was natural to label the discrete frequencies ν with natural numbers n . To fit the spectrum of a given atom, say hydrogen, let us try the ansatz

$$\nu = g_1 n_1^{q_1} + g_2 n_2^{q_2}. \quad (1)$$

We view this ansatz as a slot machine, you input two bills, that is integers q_1, q_2 and two coins, that is two real numbers g_1, g_2 and compare the output with the measured spectrum, see figure 1. For the curious reader we should explain why the integers n_j are considered more precious than the reals g_j . It is because before Balmer and Rydberg there was a complicated theory, forgotten today, called exponent quantization. This theory explained how — assuming the existence of monopoles — exponents like the ones above are necessarily integers. Anyhow, if you are rich enough you play and replay on the slot machine until you win. The winner is

the Balmer-Rydberg formula, $n_1 = n_2 = -2$, $g_1 = -g_2 = 3.289 \cdot 10^{15}$ Hz, the famous Rydberg constant R . Then came quantum mechanics. It explained why the spectrum of the hydrogen atom was discrete in the first place, derived the exponents and the Rydberg constant,

$$R = \frac{m_e}{4\pi\hbar^3} \frac{e_4}{(4\pi\epsilon_0)^2}, \quad (2)$$

from a noncommutativity, $[x, p] = i\hbar 1$.

To cut short its long and complicated history we introduce the standard model as the winner of a particular slot machine. This machine which has become popular under the names of Yang, Mills and Higgs has four slots for four bills. Once you have decided which bills you choose and entered them, a certain number of small slots will open for coins. Their number depends on the choice of bills. You make your choice of coins, feed them in, and the machine starts working. It produces as output a Lagrangian density. From this density perturbative quantum field theory allows you to compute a complete particle phenomenology: the particle spectrum with their quantum numbers, cross sections, life times, branching ratios, see figure 2. You compare the phenomenology to experiment to find out whether your input wins or loses.

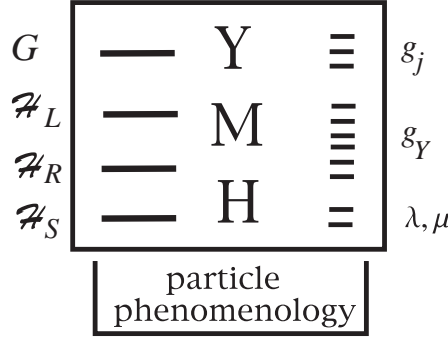


Figure 2: The Yang-Mills-Higgs slot machine

1.1 The input

The first bill is a finite dimensional, real, compact Lie group G . The gauge bosons A , spin 1, will live in its adjoint representation whose Hilbert space is the complexified of the Lie algebra \mathfrak{g} .

The remaining bills are three unitary representations of G , ρ_L , ρ_R and ρ_S , defined on the complex Hilbert spaces \mathcal{H}_L , \mathcal{H}_R , \mathcal{H}_S . They classify the left- and right-handed fermions, ψ_L and ψ_R , spin $\frac{1}{2}$, and the scalars φ , spin 0. A massless left-handed spinor has its spin parallel to its direction of propagation, anti-parallel for the right-handed one. If the two representations \mathcal{H}_L and \mathcal{H}_R are not identical, parity is broken, because a space inversion, $\vec{x} \rightarrow -\vec{x}$ reverses

the direction of propagation but leaves the spin unchanged. The group G is chosen compact to ensure that the unitary representations are finite dimensional, we want a finite number of ‘elementary particles’ according to the credo of particle physics that particles are orthonormal basis vectors of the Hilbert spaces which carry the representations. More generally, we might also admit multi-valued representations, ‘spin representations’ which would open the debate on charge quantization.

The coins are numbers, more precisely coefficients of invariant polynomials. We need gauge couplings g_j for each simple factor of \mathfrak{g} , Then we need the Higgs potential $V(\varphi)$. It is an invariant, fourth order, stable polynomial on $\mathcal{H}_S \ni \varphi$. Stable means bounded from below. For $G = SU(2) \ni u$ and the Higgs scalar in the fundamental representation, $\varphi \in \mathcal{H}_S = \mathbb{C}^2$, $\rho_S(u) = u$, we have

$$V(\varphi) = \lambda (\varphi^* \varphi)^2 - \frac{1}{2} \mu^2 \varphi^* \varphi. \quad (3)$$

The coefficients of the Higgs potential are the Higgs couplings, λ must be positive for stability. We say that the potential breaks G spontaneously if no minimum of the potential is invariant under G . In our example, if μ is positive the minimum of $V(\varphi)$ is the 3-sphere $|\varphi| = v := \frac{1}{2} \mu / \sqrt{\lambda}$. v is called vacuum expectation value and $SU(2)$ is said to break down spontaneously. On the other hand if μ is purely imaginary, then the minimum of the potential is the origin, no spontaneous symmetry breaking. Finally, we need the Yukawa couplings g_Y . They are the coefficients of the most general trilinear invariant coupling between a scalar and two fermions, $\bar{\psi}_L \varphi \psi_R$.

If the symmetry is broken spontaneously, gauge and Higgs bosons acquire masses related to gauge and Higgs couplings, fermions acquire masses related to the Yukawa couplings.

The Lagrangian has five pieces, the Yang-Mills Lagrangian, the Klein-Gordon Lagrangian, the Higgs potential, the Dirac Lagrangian and the Yukawa terms:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \overline{D_\mu \varphi} D^\mu \varphi + V(\varphi) \\ & + \bar{\psi} \not{D} \psi + g_Y \bar{\psi} \varphi \psi, \quad \psi = \psi_L \oplus \psi_R. \end{aligned} \quad (4)$$

For $G = U(1)$ the Yang-Mills Lagrangian is nothing but Maxwell’s Lagrangian, the gauge boson A is the photon and its coupling constant g is $e\epsilon_0^{-1/2}$. The Dirac Lagrangian is the special relativistic extension of Schrödinger’s Lagrangian and ψ is the wave function of the electron and positron, coupled to the electro-magnetic field A . Electro-magnetism preserves parity, $\mathcal{H}_L = \mathcal{H}_R = \mathbb{C}$, the representation being characterized by the electric charge, -1 for both the left- and right handed electron. The other three pieces are added by hand in order to give masses to the gauge bosons and to the fermions. Without spontaneous symmetry breaking such masses are forbidden by gauge invariance and parity violation.

1.2 The winner

Physicists have spent some thirty years and billions of Swiss Francs playing on the slot machine by Yang, Mills and Higgs. There is a winner, the standard model of electro-weak and strong interactions. Its bills are

$$G = SU(2) \times U(1) \times SU(3)/(\mathbb{Z}_2 \times \mathbb{Z}_3), \quad (5)$$

$$\mathcal{H}_L = \bigoplus_1^3 \left[(2, \frac{1}{6}, 3) \oplus (2, -\frac{1}{2}, 1) \right], \quad (6)$$

$$\mathcal{H}_R = \bigoplus_1^3 \left[(1, \frac{2}{3}, 3) \oplus (1, -\frac{1}{3}, 3) \oplus (1, -1, 1) \right], \quad (7)$$

$$\mathcal{H}_S = (2, -\frac{1}{2}, 1), \quad (8)$$

where (n_2, y, n_3) denotes the tensor product of an n_2 dimensional representation of $SU(2)$, an n_3 dimensional representation of $SU(3)$ and the one dimensional representation of $U(1)$ with hypercharge y : $\rho(\exp(i\theta)) = \exp(iy\theta)$. For historical reasons the hypercharge is an integer multiple of $\frac{1}{6}$. This is irrelevant: only the product of the hypercharge by its gauge coupling is measurable and we do not need multi-valued representations which are characterized by non-integer, rational hypercharges. In the direct sum, we recognize the three generations of fermions, the quarks are $SU(3)$ colour triplets, the leptons colour singlets. The basis of the fermion representation space is

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

$$\begin{matrix} u_R, & c_R, & t_R, & & & & \\ d_R, & s_R, & b_R, & e_R, & \mu_R, & \tau_R \end{matrix}$$

The parentheses indicate isospin doublets.

We recognize the eight gluons in $su(3)$. Attention, the $U(1)$ is not the one of electric charge, it is called hypercharge, the electric charge is a linear combination of hypercharge and weak isospin, parameterized by the weak mixing angle θ_w to be introduced below. This mixing is necessary to give electric charges to the W bosons. The W^+ and W^- are pure isospin states, while the Z^0 and the photon are (orthogonal) mixtures of the third isospin generator and hypercharge.

Because of the high degree of reducibility in the bills, there are many coins, among them 27 complex Yukawa couplings. Not all of them have a physical meaning. The coins can be converted into 18 physically significant, positive numbers [2], three gauge couplings,

$$g_1 = 0.3574 \pm 0.0001, \quad g_2 = 0.6518 \pm 0.0003, \quad g_3 = 1.218 \pm 0.01, \quad (9)$$

two Higgs couplings, λ and μ , and 13 positive parameters from the Yukawa couplings. The Higgs couplings are related to the boson masses:

$$m_W = \frac{1}{2}g_2 v = 80.419 \pm 0.056 \text{ GeV}, \quad (10)$$

$$m_Z = \frac{1}{2}\sqrt{g_1^2 + g_2^2} v = m_W / \cos \theta_w = 91.1882 \pm .0022 \text{ GeV}, \quad (11)$$

$$m_H = 2\sqrt{2}\sqrt{\lambda} v > 98 \text{ GeV}, \quad (12)$$

with the vacuum expectation value $v := \frac{1}{2}\mu/\sqrt{\lambda}$ and the weak mixing angle θ_w defined by

$$\sin^2 \theta_w := g_2^{-2}/(g_2^{-2} + g_1^{-2}) = 0.23117 \pm 0.00016. \quad (13)$$

For the standard model, there is a one-to-one correspondence between the physically relevant part of the Yukawa couplings and the fermion masses and mixings,

$$\begin{aligned} m_e &= 0.510998902 \pm 0.000000021 \text{ MeV}, & m_u &= 3 \pm 2 \text{ MeV}, & m_d &= 6 \pm 3 \text{ MeV}, \\ m_\mu &= 0.105658357 \pm 0.000000005 \text{ GeV}, & m_c &= 1.25 \pm 0.1 \text{ GeV}, & m_s &= 0.125 \pm 0.05 \text{ GeV}, \\ m_\tau &= 1.77703 \pm 0.00003 \text{ GeV}, & m_t &= 174.3 \pm 5.1 \text{ GeV}, & m_b &= 4.2 \pm 0.2 \text{ GeV}. \end{aligned}$$

For simplicity, we take massless neutrinos. Then mixing only occurs for quarks and is given by a unitary matrix, the Cabibbo-Kobayashi-Maskawa matrix

$$C_{KM} := \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (14)$$

For physical purposes it can be parameterized by three angles θ_{12} , θ_{23} , θ_{13} and one CP violating phase δ :

$$C_{KM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (15)$$

with $c_{kl} := \cos \theta_{kl}$, $s_{kl} := \sin \theta_{kl}$. The absolute values of the matrix elements are:

$$\begin{pmatrix} 0.9750 \pm 0.0008 & 0.223 \pm 0.004 & 0.004 \pm 0.002 \\ 0.222 \pm 0.003 & 0.9742 \pm 0.0008 & 0.040 \pm 0.003 \\ 0.009 \pm 0.005 & 0.039 \pm 0.004 & 0.9992 \pm 0.0003 \end{pmatrix}.$$

The physical meaning of the quark mixings is the following: when a sufficiently energetic W^+ decays into a u quark, this u quark is produced together with a \bar{d} quark with probability $|V_{ud}|^2$, together with a \bar{s} quark with probability $|V_{us}|^2$, together with a \bar{b} quark with probability $|V_{ub}|^2$. The fermion masses and mixings together are an entity, the fermionic mass matrix or the matrix of Yukawa couplings multiplied by the vacuum expectation value.

Let us note six intriguing properties of the standard model.

- The gluons couple in the same way to left- and right-handed fermions, the gluon coupling is vectorial, strong interaction do not break parity.
- The fermionic mass matrix commutes with $SU(3)$, the three colours of a given quark have the same mass.
- The scalar is a colour singlet, the $SU(3)$ part of G does not suffer spontaneous break down, the gluons remain massless.
- The $SU(2)$ couples only to left-handed fermions, its coupling is chiral, weak interaction break parity maximally.
- The scalar is an isospin doublet, the $SU(2)$ part suffers spontaneous break down, the W^\pm and the Z^0 are massive.
- The remaining colourless and neutral gauge boson, the photon, is massless and couples vectorially. This is certainly the most ad-hoc feature of the standard model. Indeed the photon is a linear combination of isospin which couples only to left-handed fermions and of a $U(1)$ generator, that may couple to both chiralities. Therefore only the careful fine tuning of the hypercharges in the three input representations (6-8) can save parity conservation of electro-magnetism.

Nevertheless the phenomenological success of the standard model is phenomenal: with only a handful of parameters it reproduces correctly some millions of experimental numbers. And so far the standard model is uncontradicted.

Let us come back to our analogy between the Balmer-Rydberg formula and the standard model. One might object that the ansatz for the spectrum, equation (1), is completely ad hoc, while the class of all (anomaly free) Yang-Mills-Higgs models is distinguished by perturbative renormalizability. This is true, but this property was proved [3] only years after the electro-weak part of the standard model was published [4].

By placing the hydrogen atom in an electric or magnetic field we know experimentally that every frequency ‘state’ n , $n = 1, 2, 3, \dots$ comes with n irreducible unitary representations ℓ , $\ell = 0, 1, 2, \dots, n-1$ of dimensions $2\ell + 1$. An orthonormal basis of each representation ℓ is labelled by another integer m , $m = -\ell, -\ell + 1, \dots, \ell$. This experimental fact has motivated the credo that particles are orthonormal basis vectors of unitary representations of compact groups. This credo is also behind the standard model. While $SO(3)$ has a clear geometrical interpretation, we are still looking for such an interpretation of $SU(2) \times U(1) \times SU(3)/[\mathbb{Z}_2 \times \mathbb{Z}_3]$.

We close this section with Iliopoulos’ joke from 1976 [5]. Meanwhile his joke has become hard, experimental reality:

Do-it-yourself kit for gauge models:

- 1) Choose a gauge group G .
- 2) Choose the fields of the “elementary particles” you want to introduce, and their representations. Do not forget to include enough fields to allow for the Higgs mechanism.
- 3) Write the most general renormalizable Lagrangian invariant under G . At this stage gauge invariance is still exact and all vector bosons are massless.
- 4) Choose the parameters of the Higgs scalars so that spontaneous symmetry breaking occurs. In practice, this often means to choose a negative value for the parameter μ^2 .
- 5) Translate the scalars and rewrite the Lagrangian in terms of the translated fields. Choose a suitable gauge and quantize the theory.
- 6) Look at the properties of the resulting model. If it resembles physics, even remotely, publish it.
- 7) GO TO 1.

2 Connes’ noncommutative geometry

Connes equips Riemannian spaces with an uncertainty principle. As in quantum mechanics, this uncertainty principle derives from noncommutativity.

Consider the classical harmonic oscillator. Its phase space is \mathbb{R}^2 with points labeled by position x and momentum p . A classical observable is a differentiable function on phase space, for example the total energy $p^2/(2m) + kx^2$. Observables can be added and multiplied, they form the algebra $\mathcal{C}^\infty(\mathbb{R}^2)$ which is associative and commutative. To pass to quantum mechanics, this algebra is rendered noncommutative by means of the following noncommutation relation for the generators x and p ,

$$[x, p] = i\hbar 1. \tag{16}$$

Let us call \mathcal{A} the resulting algebra ‘of quantum observables’. It is still associative, has an involution \cdot^* , the adjoint, and a unit, 1. Of course there is no space anymore of which \mathcal{A} is the algebra of functions. Nevertheless we talk about such a ‘quantum phase space’ as a space that

has no points or a space with an uncertainty relation. Indeed the noncommutation relation implies Heisenberg's uncertainty relation

$$\Delta x \Delta p \geq \hbar/2 \quad (17)$$

and tells us that points in phase space lose all meaning, we can only resolve cells in phase space of volume $\hbar/2$.

To define the uncertainty Δa for an observable $a \in \mathcal{A}$ we need a faithful representation of the algebra on a Hilbert space, i.e. an injective homomorphism $\rho : \mathcal{A} \rightarrow \text{End}(\mathcal{H})$. For the harmonic oscillator this Hilbert space is $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. Its elements are the wave functions $\psi(x)$, square integrable functions on configuration space. Finally the dynamics is defined by a self adjoint observable $H = H^* \in \mathcal{A}$ via Schrödinger's equation

$$\left(i\hbar \frac{\partial}{\partial t} - \rho(H) \right) \psi(t, x) = 0. \quad (18)$$

Usually the representation is not written explicitly. Since it is faithful no confusion should arise from this abuse. Here time is considered an external parameter, in particular time is not considered an observable. This is different in the special relativistic setting where Schrödinger's equation is replaced by Dirac's equation,

$$\not{D}\psi = 0. \quad (19)$$

Now the wave function ψ is the four component spinor consisting of left- and right-handed, particle and antiparticle wave functions. The Dirac operator is not in \mathcal{A} anymore, but $\not{D} \in \text{End}(\mathcal{H})$. It is formally self adjoint, $\not{D}^* = \not{D}$.

Connes' geometries are described by these three purely algebraic items, $(\mathcal{A}, \mathcal{H}, \not{D})$, with \mathcal{A} a real, associative, possibly noncommutative involution algebra with unit, faithfully represented on a complex Hilbert space \mathcal{H} and \not{D} is a self adjoint operator on \mathcal{H} .

Connes' geometry [6] does to spacetime what quantum mechanics does to phase space. So the first question we have to ask is: can we reconstruct Riemannian geometry from the algebraic data of the so called **spectral triple** $(\mathcal{A}, \mathcal{H}, \not{D})$. The answer is affirmative precisely in the case where the algebra \mathcal{A} is commutative. Indeed Connes' reconstruction theorem of 1996 [7] establishes a one-to-one correspondence between commutative spectral triples and Riemannian spin manifolds.

Let us try to get a feeling of the local information contained in this theorem. Besides describing the dynamics of the spinor field ψ the Dirac operator \not{D} encodes the Riemannian metric, which is the gravitational field, and the dimension of spacetime can be read from its spectrum. The square of the Dirac operator is the wave operator which in 1+2 dimensions

governs the dynamics of a drum. Remember the question ‘Can you hear the shape of a drum?’ that relates physics and mathematics in a beautiful way. This question concerns a global property of spacetime, the boundary. Can you reconstruct it from the spectrum of the wave operator? On the other hand the dimension of spacetime is a local property. It can be retrieved from the asymptotic behaviour of the spectrum of the Dirac operator for large eigenvalues. For compact spacetime M this spectrum is discrete. Let us order the eigenvalues, $\dots \lambda_{n-1} \leq \lambda_n \leq \lambda_{n+1} \dots$. Then Weyl’s spectral theorem states that the eigenvalues grow asymptotically as $n^{1/\dim M}$. To explore a local property of spacetime we only need the high energy part of the spectrum. This is in nice agreement with our intuition from quantum mechanics and motivates the name spectral triple.

Differential forms are the main local ingredient of the Yang-Mills Lagrangian. They too are reconstructed from the spectral triple using the Dirac operator. For example the 1-form da for a function a on spacetime is reconstructed as $[\not{D}, \rho(a)]$. This is again motivated from quantum mechanics. Indeed in a 1+0 dimensional spacetime da is just the time derivative of the ‘observable’ a and is associated to the commutator of the Hamilton operator with a .

Finally and most importantly for us, Einstein’s derivation of general relativity from Riemannian geometry can be extended to spectral triples. Einstein’s starting point is Newton’s equation which describes the dynamics of a point particle. Since in noncommutative geometry points lose their meaning, Connes’ starting point is the Dirac equation which describes the dynamics of a quantum particle. Thereby Connes’ derivations remains valid for all spectral triples, commutative or not. In accordance with our language from quantum mechanics, a noncommutative spectral triple is addressed as noncommutative space or noncommutative geometry. Of course we are eager to see what Einstein’s derivation becomes in a noncommutative geometry. The simplest such geometry describes a direct product of a four dimensional spacetime with a discrete space of points. In other words we are talking about Kaluza-Klein models where the transverse space is of dimension zero. Indeed one of the advantages of the description of geometry by spectral triples, commutative or not, is that continuous and discrete spaces are included in the same formalism. Connes and Chamseddine [8, 7, 9] have repeated Einstein’s derivation for these discrete Kaluza-Klein geometries. The result is absolutely amazing. Starting from the free Dirac Lagrangian alone they derive the Einstein-Hilbert Lagrangian of gravity and simultaneous they get for free the Yang-Mills Lagrangian, the Klein-Gordon Lagrangian, the Higgs potential, the covariant Dirac Lagrangian and the Yukawa terms. In other words they derive the entire slot machine of Yang-Mills-Higgs from geometry. In this geometry the Higgs scalar is a 1-form just as the gauge bosons. The latter define parallel transport in the four continuous directions of spacetime, the former defines parallel transport in the discrete direction. The Yukawa terms are the minimal couplings of the scalars, and the

scalar self coupling, λ is related to the gauge boson self coupling g^2 in the nonAbelian case. In these noncommutative spaces of discrete Kaluza-Klein type the uncertainty is transverse, in the sense that in the four dimensional Riemannian space, points are still sharp. Take for example for the transverse dimension the two-point space. Then the direct product is the ‘two sheeted universe’ consisting of two identical copies of the four dimensional Riemannian space. While each point of the Riemannian space is well localized the uncertainty is that you do not know on which copy you are. The distance between the two copies is measured by the Higgs field. The two sheeted universe [10] was one of the first noncommutative examples to exhibit spontaneous symmetry breaking.

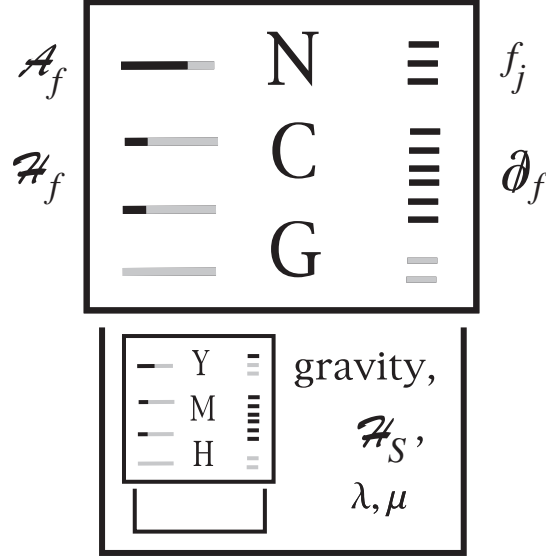


Figure 3: Connes' slot machine

Coming back to the slot machine, the only arbitrary input left is the choice of the spectral triple $(\mathcal{A}_f, \mathcal{H}_f, \partial_f)$ describing the discrete space and three constants, f_0, f_2, f_4 , figure 3. We use the index \cdot_f for finite because discrete spaces are zero-dimensional. In accordance with Weyl's theorem the algebra \mathcal{A}_f and the representation space \mathcal{H}_f are both finite dimensional. The classification of those is well known, the algebra is a sum of matrix algebras with the fundamental or singlet representations. The compact group G from the input is the group of unitaries, $U(\mathcal{A}_f)$ and the fermionic representations are $\mathcal{H}_f = \mathcal{H}_L \oplus \mathcal{H}_R$. The discrete Dirac operator ∂_f is the fermionic mass matrix, fermion masses and mixings. The Yukawa couplings remain input but they are constrained by the axioms of the spectral triple. These constraints are so tight that only very few Yang-Mills-Higgs models can be derived from noncommutative geometry as pseudo forces. No left-right symmetric model can [11], no Grand Unified Theory can, for instance the $SU(5)$ model needs a 10-dimensional fermion representations, $SO(10)$ 16-

dimensional ones, E_6 is not the group of an associative algebra. Moreover the last two models are left-right symmetric. Much effort has gone into the construction of a supersymmetric model from noncommutative geometry, in vain [12].

On the output side we find of course gravity, its cosmological constant is related to the input parameter f_0 , Newtons constant to f_2 . We find the complete Yang-Mills-Higgs Lagrangian (4). Its Higgs sector, the representation \mathcal{H}_S and the potential $V(\varphi)$, is entirely computed from the data of the finite spectral triple. The Higgs self coupling λ is related to the gauge coupling g . Both are computed from f_4 .

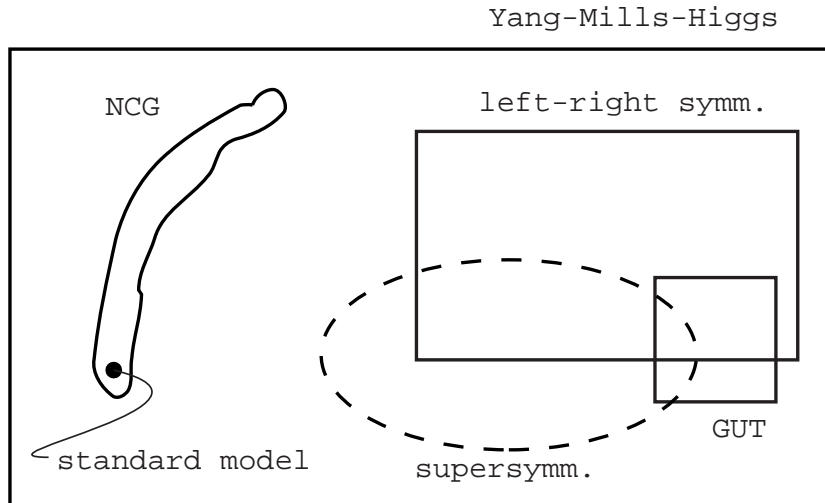


Figure 4: Pseudo forces from noncommutative geometry

The standard model fits perfectly into this frame, see figure 4. Indeed you check that its group consists of unitaries, equation (5), and that its fermionic representation consists of fundamental and singlet representations, equations (6) and (7). Furthermore the computation of the scalar representation \mathcal{H}_S yields equation (8) on the nose. Not enough, the six intriguing properties of the standard model listed in subsection 1.2 are ad hoc choices in the frame of the Yang-Mills-Higgs slot machine, they derive from the axioms of the spectral triple together with the physical assumption that parity is violated. In particular, the fermionic hypercharges can be computed and come out correctly [13]. Finally the relations among coupling constants read in the standard model,

$$g_2^2 = g_3^2 = 3\lambda. \quad (20)$$

If, like in Grand Unified Theories, we add the hypothesis of the big desert then standard renormalization flow gives a unification scale of $\Lambda = 10^{17}$ GeV where the uncertainty of spacetime is expected to become longitudinal and consequently the coupling constants should cease to run.

At the same time we get a Higgs mass of $m_H = 171 \pm 5$ GeV for a top mass of 174.3 ± 5.1 GeV.

In [14] you may find additional references on noncommutative geometry and its applications to forces. I recommend particularly the recent Costa Rica book [15].

3 Outlook

Amelino-Camelia gives three arguments [1] that the experimental observation of the uncertainty at 10^{17} GeV or 10^{-40} s might be possible within the next ten years. These observations concern local experiments on earth, like gravitational wave detectors, and measurements at cosmological scale, like γ ray bursts.

We are optimistic to be able to single out the standard model within its noncommutative frame by one physical requirement: that the spontaneous symmetry breaking on which we have no handle be such that it allow different fermion masses within one irreducible multiplet, like the left-handed top and bottom quarks, that sit in the same isospin doublet.

Noncommutative geometry reconciles Riemannian geometry and uncertainty. We expect the new paradigm, that does not recognize short distances, to clean up quantum field theory and to reconcile it with general relativity. Progress in this direction exists: Connes, Moscovici and Kreimer discovered a subtle link between a noncommutative generalization of the index theorem and perturbative quantum field theory. This link is a Hopf algebra relevant to both theories [16].

References

- [1] G. Amelino-Camelia, *Are we at the dawn of quantum gravity phenomenology?* Lectures given at 35th Winter School of Theoretical Physics: From Cosmology to Quantum Gravity, Polanica, Poland, 1999, gr-qc/9910089
- [2] The Particle Data Group, *Particle Physics Booklet* and <http://pdg.lbl.gov>
- [3] G. 't Hooft, *Renormalizable Lagrangians for massive Yang-Mills fields*, Nucl. Phys. B35 (1971) 167
G. 't Hooft & M. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. B44 (1972) 189
G. 't Hooft & M. Veltman, *Combinatorics of gauge fields*, Nucl. Phys. B50 (1972) 318

- B. W. Lee & J. Zinn-Justin, *Spontaneously broken gauge symmetries I, II, III and IV*, Phys. Rev. D5 (1972) 3121, 3137, 3155, Phys. Rev. D7 (1973) 1049
- [4] S. Glashow, *Partial-symmetries of weak interactions*, Nucl. Phys. 22 (1961) 579
 A. Salam in Elementary Particle Physics: Relativistic Groups and Analyticity, Nobel Symposium no. 8, page 367, eds.: N. Svartholm, Almqvist & Wiksell, Stockholm 1968
 S. Weinberg, *A model of leptons*, Phys. Rev. Lett. 19 (1967) 1264
- [5] J. Iliopoulos, *An introduction to gauge theories*, Yellow Report, CERN (1976)
- [6] A. Connes, *Noncommutative Geometry*, Academic Press (1994)
- [7] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, hep-th/9603053, Comm. Math. Phys. 155 (1996) 109
- [8] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194
- [9] A. Chamseddine & A. Connes, *The spectral action principle*, hep-th/9606001, Comm. Math. Phys. 186 (1997) 731
- [10] M. Dubois-Violette, R. Kerner & J. Madore, *Gauge bosons in a noncommutative geometry*, Phys. Lett. 217B (1989) 485
- [11] F. Girelli, *Left-right symmetric models in noncommutative geometry?* hep-th/0011123, Lett. Math. Phys., in press
- [12] W. Kalau & M. Walze, *Supersymmetry and noncommutative geometry*, hep-th/9604146, J. Geom. Phys. 22 (1997) 77
- [13] S. Lazzarini & T. Schücker, *A farewell to unimodularity*, hep-th/0104038, Phys. Lett. B 510 (2001) 277
- [14] A. Connes, *Essay on physics and noncommutative geometry*, in *The Interface of Mathematics and Particle Physics*, eds.: D. G. Quillen et al., Clarendon Press (1990)
 J. Madore, *Modification of Kaluza Klein theory*, Phys. Rev. D 41 (1990) 3709
 A. Connes & J. Lott, *The metric aspect of noncommutative geometry*, in the proceedings of the 1991 Cargèse Summer Conference, eds.: J. Fröhlich et al., Plenum Press (1992)
 J. C. Várilly & J. M. Gracia-Bondía, *Connes' noncommutative differential geometry and the standard model*, J. Geom. Phys. 12 (1993) 223

- J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press (1995)
- G. Landi, *An Introduction to Noncommutative Spaces and their Geometry*, hep-th/9701078, Springer, Heidelberg (1997)
- C. P. Martín, J. M. Gracia-Bondía & J. C. Várilly, *The standard model as a non-commutative geometry: the low mass regime*, hep-th/9605001, Phys. Rep. 294 (1998) 363
- J. C. Várilly, *Introduction to noncommutative geometry*, physics/9709045, to appear in the proceedings of the EMS Summer School on Noncommutative Geometry and Applications, Portugal, 1997, ed.: P. Almeida
- T. Schücker, *Geometries and forces*, hep-th/9712095, to appear in the proceedings of the EMS Summer School on Noncommutative Geometry and Applications, Portugal, 1997, ed.: P. Almeida
- D. Kastler, *Noncommutative Geometry and fundamental physical interactions: the Lagrangian level*, in the J. Math. Phys. 2000 issue: ‘Mathematical Physics: Past and Future’
- [15] J. M. Gracia-Bondía, J. C. Várilly & H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser (2000)
- [16] A. Connes & H. Moscovici, *Hopf Algebra, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. 198 (1998) 199
- D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, q-alg/9707029, Adv. Theor. Math. Phys. 2 (1998) 303
- A. Connes & D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. 1. The Hopf algebra structure of graphs and the main theorem*, hep-th/9912092, Comm. Math. Phys. 210 (2000) 249
- A. Connes & D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. 2. the beta function, diffeomorphisms and the renormalization group*, hep-th/0003188 Comm. Math. Phys. 216 (2001) 215
- for a recent review, see J. C. Várilly, *Hopf algebras in noncommutative geometry*, hep-th/010977,